EIGENFUNCTION EXPANSIONS

MAIN AIMS

TO DEVELOP OPTIMAL FORMULATIONS OF THE EIGENFUNCTION EXPANSIONS ASSOCIATED WITH THE ONE-DIMENSIONAL SCHRÖDINGER OPERATOR H ON THE REAL LINE WHICH PROVIDES A MECHANISM FOR

- IDENTIFYING A SUITABLE SCALAR SPECTRAL FUNCTION *
- ESTABLISHING THE SPECTRAL MULTIPLICITY OF H *
- EXHIBITING THE GENERALISED EIGENFUNCTIONS EXPLICITLY IN EXPANSIONS
- IDENTIFYING THE SPECTRAL TYPES OF THE EIGENFUNCTIONS FROM THEIR ASYMPTOTIC BEHAVIOUR AT $\pm\infty$.

* Already achieved by I.S. Kac, 1962/3

CONTEXT

Let L be the differential expression satisfying

- $Lu = \lambda u$, $L := -\frac{d^2}{dr^2} + q(r)$, $-\infty < r < \infty$,
- $q(r) : \mathbb{R} \to \mathbb{R} \in L_1^{loc}(\mathbb{R})$, $\lambda \in \mathbb{R}$,
- L is in Weyl's limit point case (LP) at $(\pm\infty)$ (ie. for each $z = \lambda \pm i\varepsilon$, precisely 1 solution of $Lu = \lambda u$ (the Weyl solution) of $Lu = \lambda u$ is in $L_2(\mathbb{R}^+)$.

Then a unique selfadjoint operator H (the 1-d Schrödinger operator) is defined by

$$Hf = Lf, \quad f \in \mathcal{D}(H),$$

where $f \in \mathcal{D}(H)$ if (i) $f, Lf \in L_2(\mathbb{R})$

(ii) f, f' are locally a.c. on $\mathbb{R}$

PROPERTIES OF THE SPECTRUM $\sigma(H)$

- H selfadjoint $\Rightarrow \sigma(H) = \overline{\sigma(H)} \subseteq \mathbb{R}$
- components of $\sigma(H)$ may be absolutely continuous or singular, or both
- Singular part of $\sigma(H)$, $\sigma_s(H)$, may be pure point, singular or both

  is purely singular then H has spectral multiplicity 1
- If $\sigma(H)$ includes an absolutely continuous part $\sigma_{a.c.}(H)$, then $\sigma_{a.c.}(H)$ will contain a subset S_1 with positive Lebesgue measure $|S_1|$ on which H has spectral multiplicity 1 AND/OR a subset S_2 s.t. $|S_2| > 0$ on which H has spectral mult. 2.

Terminology: Simple / degenerate spectrum refer to spectral mult. 1 / mult. 2 respectively.

EIGENFUNCTION EXPANSIONS IN ½ LINE CASE

Let H be a selfadjoint operator associated with

$$L = -\frac{d^2}{dr^2} + q(r), \quad 0 \leq r < \infty,$$

and boundary condition

$$\cos\alpha\, u(0,\lambda) + \sin\alpha\, u'(0,\lambda) = 0 \qquad (1)$$

for some fixed $\alpha \in [0,\pi)$. Then if $f \in L_2(\mathbb{R}^+)$

$$f(r) = \underset{\omega\to\infty}{\mathrm{l.i.m}} \int_{-\omega}^{\omega} u(r,\lambda)\, G(\lambda)\, d\rho_\alpha(\lambda) \qquad (2)$$

where $u(r,\lambda)$ satisfies $Lu = \lambda u$ and (1), $\rho_\alpha(\lambda)$ is spectral function,

and $$G(\lambda) = \underset{\substack{\tau\uparrow\infty \\ \nu\downarrow 0}}{\mathrm{l.i.m.}} \int_{\nu}^{\tau} u(r,\lambda)\, f(r)\, dr$$

with convergence in $L_2(\mathbb{R}^+)$ and $L_2(\mathbb{R}, d\rho_\alpha)$ respectively.

N.B. In this context, $u(r,\lambda)$ is said to be a generalised eigenfunction of H with eigenvalue $\lambda \Leftrightarrow 0 < \rho'_\alpha(\lambda) \leq \infty$.

WEYL-KODAIRA EIGENFUNCTION EXPANSION ON THE LINE

Let $f(r) \in L_2(\mathbb{R})$. Then

$$f(r) = \underset{\omega \to \infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} \sum_{i,j=1,2} u_i(r,\lambda) F_j(\lambda)\, d\rho_{ij}(\lambda),$$ where

$$F(\lambda) = \{F_1(\lambda), F_2(\lambda)\} = \underset{\nu \to \infty}{\mathrm{l.i.m.}} \left\{ \int_{-\nu}^{\nu} u_1(r,\lambda) f(r)\, dr, \int_{-\nu}^{\nu} u_2(r,\lambda) f(r)\, dr \right\}$$

with convergence in $L_2(\mathbb{R})$ and $L_2(\mathbb{R}, d\rho_{ij})$ respectively, where (ρ_{ij}) is the 2×2 spectral matrix associated with H, and $\{u_1(r,\lambda), u_2(r,\lambda)\}$ is a FSS of $Lu = \lambda u$ with $W(u_2, u_1) = 1$

NOTE: The W-K expansion has <u>very general application</u>

<u>BUT it FAILS to</u> • identify a <u>scalar spectral function</u>

• exhibit the <u>generalised eigenfunctions</u> • determine <u>multiplicity</u> of $\sigma(H)$ • enable <u>spectral type of eigenfns.</u> to be determined.

REFORMULATION OF WEYL-KODAIRA EXPANSION

Using Titchmarsh-Weyl theory, including the m-functions $m_{-\infty}, m_{\infty}$, associated with the ½ line operators $H_{-\infty}, H_{\infty}$ on $L_2((-\infty,0])$ and $L_2([0,\infty))$ respectively, the integrand in the W-K expansion can be expressed in matrix form to give

$$\int_{-\omega}^{\omega} \sum_{i,j=1} u_i(r,\lambda) F_j(\lambda)\, d\rho_{ij}(\lambda) = \int_{-\omega}^{\omega} (U(r,\lambda))^t \left(\frac{d\rho_{ij}}{d\rho_T}(\lambda) \right) F(\lambda)\, d\rho_T(\lambda) \quad (3)$$

where $U(r,\lambda) = \begin{pmatrix} u_1(r,\lambda) \\ u_2(r,\lambda) \end{pmatrix}$, $F(\lambda) = \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix}$, $\rho_T = \rho_{11} + \rho_{22}$

and $\left(\frac{d\rho_{ij}}{d\rho_T}(\lambda) \right) = \begin{pmatrix} \frac{d\rho_{11}}{d\rho_T}(\lambda) & \frac{d\rho_{12}}{d\rho_T}(\lambda) \\ \frac{d\rho_{21}}{d\rho_T}(\lambda) & \frac{d\rho_{22}}{d\rho_T}(\lambda) \end{pmatrix}$ is known as the spectral density matrix (SDM) and is positive semi-definite

Note that the trace of the spectral matrix (ρ_{ij}) now constitutes a scalar spectral function $\rho_T(\lambda)$

REFORMULATION OF W-K EXPANSION (Cont'd)

Since SDM is symmetric and PSD there exists a 2×2 matrix P s.t. $\left(\frac{d\rho_{ij}}{d\rho_T}(\lambda)\right) = P^t D P$ where $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ according as rank $\left(\frac{d\rho_{ij}}{d\rho_T}(\lambda)\right) = 1$ or 2 respectively.

Setting $PU(r,\lambda) = V(r,\lambda) = \begin{pmatrix} v_1(r,\lambda) \\ v_2(r,\lambda) \end{pmatrix}$, $PF(\lambda) = G(\lambda) = \begin{pmatrix} G_1(\lambda) \\ G_2(\lambda) \end{pmatrix}$,

and substituting into (3) yields

$$f(r) = \underset{\omega \to \infty}{\text{l.i.m.}} \left\{ \int_{-\omega}^{\omega} V(r,\lambda)^t D\, G(\lambda)\, d\rho_T(\lambda) \right.$$

$$= \underset{\omega \to \infty}{\text{l.i.m.}} \left\{ \int_{(-\omega,\omega)\cap \mathcal{M}_1} v(r,\lambda)\, G(\lambda)\, d\rho_T(\lambda) + \sum_{i=1}^{2} \int_{(-\omega,\omega)\cap \mathcal{M}_2} v_i(r,\lambda) G_i(\lambda)\, d\rho_T \right\}$$

Note:
$\mathcal{M}_k = \{\lambda \in \mathbb{R} :$ rank SDM $= k\}$

KAC THEOREMS (1962/3)

I H has multiplicity 2 $\Leftrightarrow$ the Lebesgue measure $|S_2|$ of

$$S_2 = \left\{ \lambda \in \mathbb{R} : \left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) \text{exists and has rank } 2 \right\}$$

is strictly positive.

If $|S_2| = 0$, then H has simple spectrum

II If $|S_2| > 0$, then the degenerate spectrum is purely a.c.

NOTE If $\sigma(H)$ is purely simple then the rank of the SDM is 1 for some or all $\lambda \in \mathbb{R}$, and is otherwise 0. The spectrum $\sigma(H)$ may be singular and/or absolutely continuous

GENERALISED EIGENFUNCTIONS & CONCEPT OF SUBORDINACY

Definition For a given $\lambda \in \mathbb{R}$, we say that the solution $u(r,\lambda)$ of $Lu = \lambda u$ is a generalised eigenfunction if and only if the trace spectral density $\rho'_\tau(\lambda)$ exists and $0 < \rho'_\tau(\lambda) \leq \infty$.

Comment There is a striking correlation between rank (SDM), the asymptotic behaviour of solutions of $Lu = \lambda u$ at $\pm\infty$, and the properties of $\sigma(H)$. To appreciate this we introduce the concept of subordinacy.

Definition A solution $u_s(r,\lambda)$ of $Lu = \lambda u$, $\lambda \in \mathbb{R}$, $0 \leq r < \infty$, is said to be subordinate at ∞ if $\lim_{N\to\infty} \left(\|u_s(r,\lambda)\|_N / \|u(r,\lambda)\|_N \right) = 0$ where $\|\cdot\|$ denotes $L_2[0,N]$ norm, and $u_s(r,\lambda)$ is linearly independent from $u(r,\lambda)$.

GENERALISED EIGENFUNCTIONS (Cont'd)

THEOREM: For a given $\lambda \in \mathbb{R}$ a solution $u(r,\lambda)$ of $Lu = \lambda u$ is a generalised eigenfunction of H $\Leftrightarrow$ one of the following alternatives holds:

(i) $u(r,\lambda)$ is a subordinate solution of $Lu = \lambda u$ at ∞, but no solution is subordinate at $-\infty$.

(ii) as in (i), but with ∞ and $-\infty$ interchanged.

(iii) the solution $u(r,\lambda)$ is subordinate at both $-\infty$ and ∞.

(iv) No solution $u(r,\lambda)$ is subordinate at ∞ or $-\infty$.

In cases (i) – (iii), the generalised eigenfunctions are unique distinguished solutions of $Lu = \lambda u$ and are correlated with the simple spectrum of H. In case (iv), all solutions of $Lu = \lambda u$ are of comparable asymptotic size at $\pm\infty$ and such values of λ are associated with the degenerate spectrum of H (i.e. mult. 2)

Reference

Daphne J. Gilbert

Eigenfunction Expansions associated with the One Dimensional Schrödinger Operator

in Operator Theory: Advances and Applications
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(includes detailed derivations and worked examples)